TWO PROPOSITIONS INVOLVING THE STANDARD REPRESENTATION OF S_n

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ABSTRACT. We present here two standalone results from a forthcoming work on the analysis of Markov chains using the representation theory of S_n. First, we give explicit formulas for the decompositions of tensor powers of the defining and standard representations of S_n. Secondly, we prove that any Markov chain on S_n starting with one fixed point and whose increment distributions are class measures will always average exactly one fixed point.

The defining, or permutation, representation of S_n is the n-dimensional representation ϱ where

$$(\varrho(\sigma))_{i,j} = \begin{cases} 1 & \sigma(j) = i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Since the fixed points of σ can be read off of the matrix diagonal, the character of ϱ at σ, $\chi_\varrho(\sigma)$, is precisely the number of fixed points of σ. The irreducible representations of S_n are parametrized by the partitions of n, and ϱ decomposes as $S^{(n-1,1)} \oplus S^{(n)}$. Note that $\chi_{S^{(n-1,1)}}(\sigma)$ is one less than the number of fixed points of σ. In the terminology of [FH91], we call the $(n-1)$-dimensional irrep $S^{(n-1,1)}$ the standard representation of S_n.

Our first proposition gives a nice formula for the decomposition of tensor powers of ϱ into irreps, i.e. the coefficients $a_{\lambda,r}$ in the expression

$$\varrho^{\otimes r} = \bigoplus_{\lambda \vdash n} a_{\lambda,r} S^\lambda := \bigoplus_{\lambda \vdash n} (S^\lambda)^{\oplus a_{\lambda,r}}. \tag{2}$$

Proposition 1. *Let $\lambda \vdash n$ and $1 \leq r \leq n - \lambda_2$. The multiplicity of S^λ in the irreducible representation decomposition of $\varrho^{\otimes r}$ is given by*

$$a_{\lambda,r} = f^{\bar\lambda} \sum_{i=|\bar\lambda|}^{r} \binom{i}{|\bar\lambda|} \left\{ {r \atop i} \right\}, \tag{3}$$

where $\bar\lambda = (\lambda_2, \lambda_3, \ldots)$ with weight $|\bar\lambda|$, $f^{\bar\lambda}$ is the number of standard Young tableaux of shape $\bar\lambda$, and $\left\{ {r \atop i} \right\}$ is a Stirling number of the second kind.

Proof. The heavy lifting had already been done by Goupil and Chauve, who derived in [GC06] the generating function

$$\sum_{r \geq |\bar\lambda|} a_{\lambda,r} \frac{x^r}{r!} = \frac{f^{\bar\lambda}}{|\bar\lambda|!} e^{e^x - 1} (e^x - 1)^{|\bar\lambda|}. \tag{4}$$

By (24b) and (24f) in Chapter 1 of [Sta97],

$$\sum_{s \geq j} \left\{ {s \atop j} \right\} \frac{x^s}{s!} = \frac{(e^x - 1)^j}{j!} \tag{5}$$

and

$$\sum_{t\geq 0} B_t \frac{x^t}{t!} = e^{e^x-1}, \tag{6}$$

where $B_0 := 1$ and $B_t = \sum_{q=1}^{t} \left\{ {t \atop q} \right\}$ is the t-th Bell number, so we obtain from (4) that

$$\frac{a_{\lambda,r}}{r!} = f^{\bar\lambda} \sum_{s+t=r} \frac{B_t}{s!t!} \left\{ {s \atop |\bar\lambda|} \right\}, \tag{7}$$

and thus

$$\begin{aligned} \frac{a_{\lambda,r}}{f^{\bar\lambda}} &= \sum_{t=0}^{r-|\bar\lambda|} B_t \binom{r}{t} \left\{ {r-t \atop |\bar\lambda|} \right\} \\ &= \left\{ {r \atop |\bar\lambda|} \right\} + \sum_{t=1}^{r-|\bar\lambda|} \sum_{q=1}^{t} \left\{ {t \atop q} \right\} \binom{r}{t} \left\{ {r-t \atop |\bar\lambda|} \right\} \\ &= \left\{ {r \atop |\bar\lambda|} \right\} + \sum_{q=1}^{r-|\bar\lambda|} \sum_{t=q}^{r-|\bar\lambda|} \left\{ {t \atop q} \right\} \binom{r}{t} \left\{ {r-t \atop |\bar\lambda|} \right\}. \end{aligned} \tag{8}$$

By (24.1.3, II.A) of [AS65],

$$\sum_{t=q}^{r-|\bar\lambda|} \left\{ {t \atop q} \right\} \binom{r}{t} \left\{ {r-t \atop |\bar\lambda|} \right\} = \binom{q+|\bar\lambda|}{|\bar\lambda|} \left\{ {r \atop q+|\bar\lambda|} \right\}, \tag{9}$$

so that

$$\begin{aligned} \frac{a_{\lambda,r}}{f^{\bar\lambda}} &= \left\{ {r \atop |\bar\lambda|} \right\} + \sum_{q=1}^{r-|\bar\lambda|} \binom{q+|\bar\lambda|}{|\bar\lambda|} \left\{ {r \atop q+|\bar\lambda|} \right\} \\ &= \left\{ {r \atop |\bar\lambda|} \right\} + \sum_{i=|\bar\lambda|+1}^{r} \binom{i}{|\bar\lambda|} \left\{ {r \atop i} \right\} = \sum_{i=|\bar\lambda|}^{r} \binom{i}{|\bar\lambda|} \left\{ {r \atop i} \right\}, \end{aligned} \tag{10}$$

as was to be shown. □

Now, let $b_{\lambda,r}$ be the multiplicities such that

$$(S^{(n-1,1)})^{\otimes r} = \bigoplus_{\lambda\vdash n} b_{\lambda,r} S^\lambda. \tag{11}$$

Goupil and Chauve also derived the generating function

$$\sum_{r\geq|\bar\lambda|} b_{\lambda,r} \frac{x^r}{r!} = \frac{f^{\bar\lambda}}{|\bar\lambda|!} e^{e^x-x-1}(e^x-1)^{|\bar\lambda|}, \tag{12}$$

so from Proposition 1 we can obtain a formula for the decomposition of $(S^{(n-1,1)})^{\otimes r}$ as well.

Corollary 1. *Let $\lambda \vdash n$ and $1 \le r \le n - \lambda_2$. The multiplicity of S^λ in the irreducible representation decomposition of $(S^{(n-1,1)})^{\otimes r}$ is given by*

$$b_{\lambda,r} = f^{\bar{\lambda}} \sum_{s=|\bar{\lambda}|}^{r} (-1)^{r-s} \binom{r}{s} \left(\sum_{i=|\bar{\lambda}|}^{s} \binom{i}{|\bar{\lambda}|} \left\{ \begin{matrix} s \\ i \end{matrix} \right\} \right). \tag{13}$$

Proof. Comparing (12) with (4) gives

$$\sum_{r \ge |\bar{\lambda}|} b_{\lambda,r} \frac{x^r}{r!} = \left(\sum_{s \ge |\bar{\lambda}|} a_{\lambda,s} \frac{x^s}{s!} \right) e^{-x} = \left(\sum_{s \ge |\bar{\lambda}|} a_{\lambda,s} \frac{x^s}{s!} \right) \left(\sum_{t \ge 0} \frac{(-x)^t}{t!} \right), \tag{14}$$

so that

$$\frac{b_{\lambda,r}}{r!} = \sum_{s+t=r} \frac{(-1)^t a_{\lambda,s}}{s!t!} = \sum_{s=|\bar{\lambda}|}^{r} \frac{(-1)^{r-s}}{s!(r-s)!} \left(f^{\bar{\lambda}} \sum_{i=|\bar{\lambda}|}^{s} \binom{i}{|\bar{\lambda}|} \left\{ \begin{matrix} s \\ i \end{matrix} \right\} \right), \tag{15}$$

and the result follows. $\square$

Remark. Corollary 1 is very similar to Proposition 2 of [GC06], but our result is slightly cleaner, as it does not involve associated Stirling numbers of the second kind.

For our second proposition, we use $S^{(n-1,1)}$ to prove a martingale-like property about the number of fixed points for certain Markov chains on S_n. Before doing so, however, some preliminaries are in order.

Let μ be a measure on a finite group G. The Fourier transform of μ is a matrix-valued map on the irreps of G defined by $\hat{\mu}(\rho) = \sum_{g \in G} \mu(g)\rho(g)$. The convolution of two measures μ and ν on G is the measure defined by

$$\mu * \nu = \sum_{h \in G} \mu(gh^{-1})\nu(h), \tag{16}$$

and the Fourier transform transforms convolutions to pointwise products: $\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}$. If μ is a class measure, then for every irrep ρ of G, we have that

$$\hat{\mu}(\rho) = \left(\frac{1}{d_\rho} \sum_{g} \mu(g)\chi_\rho(g) \right) I_{d_\rho}, \tag{17}$$

where d_ρ is the dimension of ρ. For a detailed introduction to non-commutative Fourier analysis in the context of Markov chain theory, see Chapter 16 of [Beh00].

Let E_μ denote expectation with respect to μ, and let ρ be an irrep of S_n, then as observed in Chapter 3D of [Dia88],

$$E_\mu(\chi_\rho) = \sum_{\sigma \in S_n} \mu(\sigma)\mathrm{tr}(\rho(\sigma)) = \mathrm{tr} \left(\sum_{\sigma \in S_n} \mu(\sigma)\rho(\sigma) \right) = \mathrm{tr}(\hat{\mu}(\rho)). \tag{18}$$

We can now state and prove the proposition, which says that if a Markov chain on S_n whose increment distributions are class measures starts with one fixed point, then it will always average exactly one fixed point.

Proposition 2. *Form Markov chain $\{X_i\}$ on S_n as follows: let X_0 be the identity, and set $X_1 = \tau_1 X_0$, where τ_1 is selected according to any class measure supported on the set of permutations with one fixed point. For $k \geq 2$, set $X_k = \tau_k X_{k-1}$, where τ_k is selected according to any class measure on S_n (the measure can be different for each k). Then the expected number of fixed points of X_k is one for all $k \geq 1$.*

Proof. Let ν_1 be a class measure supported on the set of permutations with one fixed point, $\nu_2, \nu_3, \ldots, \nu_k$ be class measures on S_n, and define $\mu_k = \nu_k * \cdots * \nu_2 * \nu_1$. By (18),

$$E_{\mu_k}(\chi_{S^{(n-1,1)}}) = \operatorname{tr}[\widehat{\mu_k}(S^{(n-1,1)})] = \operatorname{tr}[\widehat{\nu_1}(S^{(n-1,1)})\widehat{\nu_2}(S^{(n-1,1)}) \cdots \widehat{\nu_k}(S^{(n-1,1)})], \tag{19}$$

where

$$\widehat{\nu_1}(S^{(n-1,1)}) = \left(\frac{1}{n-1} \sum_{\sigma \in S_n} \nu_1(\sigma) \chi_{S^{(n-1,1)}}(\sigma) \right) I_{n-1} \tag{20}$$

by (17). Consider the anatomy of the partition $(n-1, 1)$: under the Murnaghan-Nakayama rule (see Theorem 4.10.2 of [Sag10]), the only way for a single box to remain at the end is for the box in the second row to have been removed as a singleton, which requires a cycle type with at least two fixed points. This means that $\chi_{S^{(n-1,1)}}(\sigma) = 0$ if σ has one fixed point. On the other hand, if σ does not have exactly one fixed point, then $\nu_1(\sigma) = 0$. Thus $\widehat{\nu_1}(S^{(n-1,1)}) = \mathbf{0}$, which in turn implies that $E_{\mu_k}(\chi_{S^{(n-1,1)}}) = 0$, and hence the expected number of fixed points with respect to μ_k is one for all $k \geq 1$. $\square$

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